An Elementary Treatise on the Application of the Algebraic Analysis to Geometry

Wesley Stoker Barker Woolhouse

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The decidedly great advantage of the Modern Mathematicians over the Ancients, has almost entirely arisen from the introduction and refinement of the Algebraic Analysis, united with the Differential and Integral Calculus; and particularly from the truly elegant and systematic mode which has been adopted in their application to problems connected with Geometry.

AN ELEMENTARY TREATISE ON THE APPLICATION OF THE ALGEBRAIC ANALYSIS TO GEOMETRY

By

Wesley Stoker Barker Woolhouse
PREFACE

The decidedly great advantage of the Modern Mathematicians over the Ancients, has almost entirely arisen from the introduction and refinement of the Algebraic Analysis, united with the Differential and Integral Calculus; and particularly from the truly elegant and systematic mode which has been adopted in their application to problems connected with Geometry.

It is but recently that the plan of determining the relative geometrical positions of points, by means of their ordinates related to fixed axes, has engaged the attention, generally, of our English writers. By means of this simple and ingenious contrivance, almost an unlimited power has been acquired in the resolution of physical problems, which require the aid of Geometry, and the differential and integral calculus has been employed with surprising efficacy. In many cases, wherein the greatest effort of the imagination would be inadequate to afford any clear or correct reasonings, the relations of the figure and the necessary conditions can be fully and distinctly put down by means of analytical equations; and the several operations, which are generally attended with much facility, depend, in a great measure, on the general principles of elimination.

The following concise introductory treatise was commenced with a view of setting forth, in the most simple and perspicuous manner, the principal formulae which may be found useful in the solutions of the various descriptions of geometrical and physical problems; and furthermore to condense them into one small volume, with a progressive arrangement, so as afford the utmost facility in referring to such may be applicable to any particular subject.
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THE MODERN APPLICATION OF THE ALGEBRAIC ANALYSIS TO GEOMETRY.

SECTION I.

DEFINITIONS AND FIRST PRINCIPLES.

ARTICLE 1. *A Function* is any analytical expression, containing one or more variable quantities, combined or not with constant quantities; it is called a function of the variable quantity, or quantity, which it contains. Thus \( x^2 + ax, (a^2 - x^2) \) are algebraic functions of \( x \); \( \tan^{-1}x \) which denotes the length of the circular arc whose radius is unity and tangent \( x \), is a trigonometric function of \( x \); and \( x + y^2 + xy, ax + by - \frac{x^2y}{c} \) are algebraic functions of \( x \) and \( y \).

2. It hence appears that functions of any quantities have values entirely dependent on the values of those quantities. For when the values of any quantities are given, the values of any functions in which they may be involved become immediately determinable.
3. A function of any variable or variable is generally denoted by prefixing one of the characters F,f,f,Y, &c Sometime functions of single variables are distinguished by their capital; thus X,X’,X” &c being taken to represent functions of x and Y,Y’,Y”,&c. functions of y.

4, Functions are said to be the same when the variable quantity, or quantities, which they contain, enter into their respective expressions in exactly the same manner. Thus \(x^2 + ax\) is the same function of \(x\) that \(y^2 + ay\) is of \(y\), and \(x^2 + y^2 + xy\) is the same function of \(x\) and \(y\) that \(u^2 + v + u\) \(v\) is of \(u\) and \(v\) Supposing \(j\) to denote the characteristic of \(x^2 + ax\), that is, supposing \(x^2 + ax\) to be indicated by \(j x\), the expression \(y^2 + ay\) will thence be indicated by \(j y\); Similarly if \(x^2 + y^2 + ay\) be represented by \(f (x, y)\), the value of \(u^2 + v^2 + uv\) will hence be denoted by \(f (u,v)\). Also assuming \(y(x,y)\) for \(ax + by - \frac{x^2 y}{c}\) the same transfer of notation \((u, v)\) will denominate the expression \(au+bv - u^2v/c\).

5 In order to express geometrical positions algebraically, points are referred to what are called the axes of co-ordinates.
These axes of co-ordinates are two fixed indefinite right lines $OA$, $OB$, taken on a given plane and parallel to two given lines so as to form a given angle. $AOB$.

The point $O$ from whence they proceed is called the origin. if any point assumed in the same plane and the parallelogram $PD'OD$ completed, the portions $OD$, $OD'$, taken from the origin along the axes, are called the co-ordinates of the point Preferred to the axes $OA$ and $OB$ These ordinates are usually denoted by $x$ and $y$, viz: $OD$ by $x$ and $OD'$ or $DP$ by $y$: and $OA$ is hence called the axis of $x$ and $OB$ the axis of $y$.

6. When the axes $OA$, $OB$ are perpendicular to each other, they are called ‘rectangular axes; and $OD$, $DP$ are then called the rectangular co-ordinates of the point $P$.

7. If the position of a point be on the contrary side of the axis $OA$ its ordinate $y$ will have a negative value; and if it be on the contrary side of $OB$, or if the point $D$ be on the contrary side of the origin $O$, we shall similarly have its ordinate $x$ negative. For the ordinates are then estimate from the origin in the opposite directions.
8. Particular ordinate which appertain to fixed points are distinguished thus, \(x' y' x'' y''\), &c.; and the points thus determined are called the points \(x' y', x'' y''\), &c.. A point on \(O A\) has \(y' = 0\) and is denoted by \(x' o\), and a point on \(O B\) is denoted by \(y' o\).

9. If the ordinates \(x\) and \(y\) are so related as to fulfill a given equation in which they are involved with constants, we shall have particular values of \(y\) for each particular value of \(x\).

Thus let

\[ f(x,y) = 0 \]

be the equation which connects the ordinates \(x\) and \(y\), \(f(x,y)\) denoting some given function. Then if a series of ordinates \(O D, O D'\) &c. be assumed as values of \(x\), are the respectively corresponding values of \(y\) will be determined by the solutions of the proposed equation; and the point \(P\) is hence limited to a certain curve line \(PP'\). This curve line is said to be represented or indicated by the given equation, which is usually called the *equation of the curve*.
10. Rectangular axes are more extensively used, and, in most cases, are more simple in their application than those which are oblique. Sometimes, however, in particular geometrical investigations, the axes may, with great advantages be taken parallel to certain given lines in the figure; yet, as rectangular axes are generally of more utility, we shall confine ourselves to the consideration of them in the subsequent investigations.

SECTION II.

EQUATIONS OF THE FIRST DEGREE.

11. The equation of the first degree is of the form
\[ a x + b y + c = 0, \ldots \ (1). \]

wherein each of the constants \(a, b, c\), may be either positive or negative.

By solving for \(y\), it gives
\[ y = -\frac{a}{b} x - \frac{c}{b} \]
and hence, assuming
\[ m = -\frac{a}{b}, h = -\frac{c}{b} \]
it becomes
y=m x+h....(2).

This form of equation, in which the constants m and h may be either positive or negative, evidently then comprehends in it all cases whatever of the first degree, the same as the above equation marked (1).

12. Let C’ be any assumed invariable point in the locus determined by the equation

\[ y=mx + h, \]

whose ordinates \( OG = x' \); \( GC' = y' \) will hence satisfy the equation

\[ y' = mx' + h. \]

Let also \( P \) be any other point in the locus whose ordinates \( OD, DP \) fulfil the equation.

\[ y= mx + h; \]

and, by deducting the above, we derive

\[ y - y' = m (x'-x'). \]

But, by drawing \( C'D'H' \) parallel to \( O A \) the axis of \( x \), it is clearly seen that \( x - x' = GD = c'D' \) and \( y-y' = D'P \); therefore
\[ D'P = m \, C \, D' \]

and \( m = \frac{D'P}{C'D'} = \tan DPC' \)

Since this property applies equally to all points \( P \) whose ordinates fulfil the proposed equation, the determined locus is evidently a straight line the points \( C' \) making an angle with the axis of \( x \) whose tangent = \( m \).

Similarly the equation
\[ a \, x + b \, y + c = 0 \]
determines a right line intersecting the axis of \( x \) at an angle whose tangent = \( -\frac{a}{b} \), (see article 11).

13. It hence appears that all equations of the first degree determine straight lines; also, (11,) that these equations may be reduced to either of the forms
\[ a \, x + b \, y + c = 0 \ldots (1,) \]
\[ y = m \, x + b \ldots (2,) \]

The latter of these is the more simple in its application, in consequence of its involving only two constants, \( m, h \). However, as equations will not always reduce to this form with-
out fractions, we shall generally deduce the several formula for both cases. And it may be observed that formula applying to the former may be easily rendered suitable latter by substituting

$m$ for $a$; $-1$ for $b$, and $h$ for $c$,

as the equation would then become

$m \times -y + h = 0$

or $y = m \times + h$,

agreeing with the latter equation.

It may also be observed that, *vice versa*, formula belonging latter mode of equation may be appropriated former by substituting, (11)

$-\frac{a}{b}$ for $m$, and $-\frac{c}{b}$ for $h$

14. Since, (12,) $m$ is the tangent of the angle which the line make with the axis of $x$, by calling this angle $w$, the ‘equation of the line may be expressed thus:

$y = x \tan w + h$. 
15. When \( y = o \), the point \( P \) must obviously be situated on the axes of \( x \), and will therefore determine the intersection of the line with \( O A \). Put \( y = o \) in the equation

\[ ax + by + c = o \]

and let \( x'' \) be the corresponding value of \( x \) and we shall thence have

\[ x'' = \frac{-c}{a}, \]

that is,

\[ OI = \frac{-c}{a}, \]

\( I \) being the intersection of the line with the axis of \( x \). Similarly, by taking \( x = o \), we find the intersection on the axis of \( y \) to be determined by

\[ y'' = \frac{-b}{c}. \]

16. By dividing the equation of the line by \( c \) and changing the signs, it becomes

\[ \frac{-a}{b} x - \frac{b}{c} y - 1 = 0 \]

which is hence equivalent to

\[ \frac{x}{x''} + \frac{y}{y''} - 1 = 0 \]
or \( \frac{x}{x''} + \frac{y}{y''} = 1 \)

This equation may therefore be regarded as determining that straight line which cuts portions from off the axes of \( x \) and \( y \), estimated from the origin respectively equal to \( x'' \) and \( y'' \).

17. When \( c=0 \), or \( h = 0 \), the equation becomes of the form

\[ ax + by = 0 \]

or \( y = mx \),

which is satisfied with \( x = 0, y = 0 \).

This shews the line to pass through the origin. In this case we have \( x''=0, y''=0 \).

18. When \( a = 0 \), the equation is

\[ by + c = 0 \]

or \( y = -\frac{c}{b} \), a constant value.

In this case the value of \( x \) is arbitrary; and therefore the line is parallel to the axis of at the distance \( -\frac{c}{b} \) on the side where \( y \) is positive.

if \( b = 0 \), or the equation be of the form
a x + c = 0,

we shall have

\[ x = -\frac{c}{a} \quad \text{and } y \text{ arbitrary}; \]

and consequently the line is parallel to the axis of \( y \) at the
distance \(-\frac{c}{a}\) on the side where \( x \) is positive.

19. When the equation becomes simply \( b y = 0 \), or \( y := 0, \)
it evidently determines the axis of \( x \); and similarly when it is \( x = 0 \) it indicates the axis of \( y \).

20. We have seen, (12,) that the tangents of the angles

which the straight lines

\[ a x + b y + c = 0 \]

\[ y = mx + h \]

make with the axis of \( x \) are respectively \(-\frac{a}{b}\) and \( m \).

It therefore appears that equations, which have the values
of \( a/b \), or of \( m \), the same, determine parallel lines. Thus

\[ ax + by + c = 0, \]

\[ ax + by + c' = 0, \]

\[ nax + nby + c'' = 0 \]
represent parallel lines.

And similarly, the equations

\[ y = mx + h, \]

\[ y = m'x + h' \]

determine parallel lines.

SECTION III.

FORMULA, &c. RELATING TO STRAIGHT LINES.

21. To express the equation of a straight line which shall pass through a given point.

As one condition is here imposed upon the line, one of the constants \( a, b, c \) will become a function of the other two and the invariable ordinate \( x'y' \) of the given point, since they have to satisfy the condition.

\[ a x' + by + c = 0 \]

Hence eliminating \( c \), by taking its value — \( (a x' + b y') \), the equation of the line, \( a x + b y + c = 0 \), becomes

\[ ax + by - (a x' + b y') = 0, \]

or \( a(x - x') + b(y - y') = 0, \)
which determines a straight line passing through the proposed point \(x'y'\).

Or it may evidently be expressed thus:

\[ y - y' = m(x-x'). \]

\(m\) being the tangent of the inclination of the line with the axis of \(x\).

22. Cor. It hence appears that the equation

\[ y - y' = (x-x') \tan \omega \]

determine a straight line passing through the point \(x'y'\) and making the angle \(\omega\) with the axis of \(x\).

23. To, determine the equation of a straight line passing through two given points.

Let \(x'y', x''y''\) be the co-ordinates of the two points; as these points are situated in the line, the constants \(m, h\), must evidently fulfil the equations

\[ y' = mx' + h, \]

\[ y'' = mx'' + h; \]

from which we find
\[ m = \frac{y'' - y'}{x'' - x'} = h = \frac{x''y' - x'y'}{x'' - x} \]

As the line passes through both of the points \( x'y', \ x''y'' \) its equation, (21,) is hence

\[ y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \]

or \( y - y'' = \frac{y'' - y'}{x'' - x'}(x - x') \)

wherein \( x'y', \ x''y'' \) are the ordinates of the two given points and \( x\ y \) points whatever in the line.

24. Given the equations of two straight lines to find the co-ordinates of the points where they intersect.

Let \( abc, \ a'b' c' \) be the constants which belong to their respective equations; and let \( x \ y \) be the ordinates of the point of intersection; this point being posited in both lines, its ordinates \( x \ y \) must satisfy both of the equations.

\[ a \ x + b \ y + c = 0, \]
\[ a' \ x + b' \ y + c' = 0. \]

From these we find

\[ x = \frac{bc' - cb'}{ab' - ba}, \text{ and } y = -\frac{ac' - ca'}{ab - ba}. \]
which determine the required point.

If the equations of the lines be of the forms

\[ y = mx + h, \]

\[ y = m'x + h' \]

we shall have

\[ x = \frac{h - h'}{m - m'}, \quad y = \frac{-m'h - mh'}{m - m'} \]

25. The co-ordinates of two points being given to find an expression for their distance, or the length of the line which joins them.

Let \( x, y, x', y' \) be the given ordinate; then drawing lines from the two points parallel to the axis of \( x \) and the other parallel to the axis of \( y \), we shall obviously have a right angled triangle whose legs are \( x - x'y - y' \); and thus the square of the required line is found to

\[ (x - x')^2 + (y - y')^2 \]

Or \( l = \sqrt{(x - x')^2 + (y - y')^2} \)
26. *The equation of a straight line being given to express its inclination with the axis of x.*

The equation of the line being

\[ ax + by + c = 0, \]

its inclination \( w \) with the axis of \( x \) is, \((12,)\) determined by

\[
\tan w = -\frac{a}{b}
\]

which gives

\[
\cos w = \frac{1}{\sqrt{1 + \tan^2 w}} = -\frac{b}{\sqrt{a^2 - b^2}}
\]

\[
\sin w = \tan w \frac{w}{\sqrt{1 + \tan^2 w}} = \frac{a}{\sqrt{a^2 + b^2}}.
\]

and

27. When the equation of the line is of the form

\[ y = mx + h, \]

we have, \((14,)\)

\[
\tan w = m \quad \text{and} \quad \cos w = \frac{1}{\sqrt{1 + m^2}}, \sin w = \frac{m}{\sqrt{1 + m^2}}
\]

28. Cor. Let \( x'' \) determine the intersection of the line with the axis of \( x \), as in article 15, and we shall evidently have, for the perpendicular from the origin on the proposed line, \( p = x'' \sin w \). Hence, substituting the value of \( x'' \), \((article 15,)\) we have
Given the equations of two right lines to find their angle of inclination.

Suppose \(a b c, a' b' c'\) to be the constants contained in their equations and let \(i\) denote the required inclination; also let, \(w, w'\) be the inclination of the two lines with the axis of \(x\).

Then, (12,)

\[
\tan w = -\frac{a}{b}, \tan w' = -\frac{a'}{b'}, \quad \tan i = -\frac{a}{b} \quad \text{and we evidently have}
\]

\[
i = w - w' \quad \text{and} \quad \tan i = \frac{\tan w - \tan w'}{1 + \tan w \tan w'}
\]

Therefore by substitution

\[
\tan i = a'b - \frac{b'a}{aa} + b' b'
\]

Hence also

\[
\cos i = \frac{1}{\sqrt{(1 + \tan^2 i)}} = \frac{aa' + bb'}{\sqrt{\{a^2 + b^2\}\{a'^2 + b'^2\}}}
\]

\[
\sin i = \cos i \cdot \tan i = \frac{a'b - b'a}{\sqrt{\{a^2 + b^2\}a'^2 + b'^2}}
\]
30. Cor. 1. If the lines be parallel, $i=0$ and $\sin i=0$;

\[
\therefore a' b - b' a = 0
\]

or \[
\frac{a'}{b'} = \frac{a}{b'},
\]

agreeing with (20.)

31. Cor. 2. When the lines are perpendicular, $\cos i = 0$

and hence

\[
a a' + b b' = 0.
\]

32. Cor. 3. If the equations of the lines be

\[
y = m x + b,
\]

\[
y = m' x + b',
\]

we shall have

\[
\tan \omega = m, \tan \omega' = m';
\]

. F

\[
\tan i = (m - m')/(1 + m m'),
\]

\[
\cos i = \frac{1 + mm'}{\sqrt{(1 + m^2)(1 + m'^2)}}
\]

\[
\sin i = \frac{m - m'}{\sqrt{(1 + m^2)(1 + m'^2)}}
\]
When the lines are parallel, \( \sin i = 0 \),

\[ m = m' \]

And when they are perpendicular, \( \cos i = 0 \), and

\[ 1 + m m' = 0. \]

33. To determine the equation of a straight line, inclined at a given angle with a given straight line. In the last proposition, article 29, let \( a \ b \ c \) be the constants to the equation of the given line and \( a' \ b' \ c' \) those of the one required. Then

\[
\tan w' = \tan(w - i) = -\frac{\tan i - \tan w}{1 + \tan w \tan i}
\]

or, substituting \(-\frac{a}{b}, -\frac{a'}{b'}\) for \( \tan w, \tan w' \),

\[
\frac{a'}{b'} = \frac{a + b \tan i}{b - a \tan i} = \frac{a \cos i + b \sin i'}{b \cos i - a \sin i'}
\]

Hence the required equation is \((a \cos i + b \sin i) x + (b \cos i - a \sin i) y + c' = 0\),

wherein \( c' \) is an indeterminate constant whose value may be found from another condition.

34. Cor. 1. If the line be required to pass through a given point \( x' y' \) its equation, (21) will therefore be
\[(a \cos i + b \sin i) (x-x')+(b \cos i-a \sin i) (y-y')=0.\]

35. Cor. 2. When the line is perpendicular to the proposed one its equation will be

\[b x - a y + c''=0.\]

36. Cor. 3. When the line is perpendicular to the given one and also passes through a given point \(x'\) \(y'\). its equation will hence be

\[b (x - x') - a (y - y') = 0.\]

The general equation of a straight line which is perpendicular to one determine by the equation

\[y=m x + h\]

is \[y = -\frac{1}{m} (x + h');\]

and, when passing through a given point \(x'y'\) is

\[y - y' = -1/ m(x - x'),\]

37. To express the length of the perpendicular on a given line from a given point.
Let \( ax + by + c = 0 \) be the equation of the line and \( x'y' \) the ordinates of the given point. Then, (36,) the equation of the perpendicular is

\[
b (x - x') - a(y - y') = 0,\]

from which, together with the above equation of the given line, we find the point where it is intersected by this perpendicular to be determined by ordinates \( x, y \) whose values are

\[
x = \frac{b2x' - a by' - ac}{a2 + b2}, \quad y = \frac{a2y' - a bx' - b c}{a2 + b2}
\]

Now, (25,) the length of the perpendicular

\[
P = \sqrt{\left( (x - x')^2 + (y - y')^2 \right)}
\]

Hence by substitution

\[
x - x' = -\frac{a(ax' + by' + c)}{a2 + b2}, \quad y - y' = -\frac{b(ax' + by' + c)}{a2 + b2} \quad \text{and}
\]

\[
P = -\frac{ax' + by' + c}{a2 + b2}.
\]

If the equation of the line be of the form \( y = mx + h \)
substituting \( m \) for \( a \), \(-1\) for \( b \) and \( h \) for \( c \), (see 13,) and we shall have

\[
p = (mx^{\wedge} + h) - y / (?1 + m^2),
\]
38. Cor. 1. When the given point is the origin, we have $x' = 0, y' = 0$; and hence, for the perpendicular from the origin on the given line, we have

$$P = -\frac{c}{\sqrt{a^2 + b^2}},$$

or

$$P = -\frac{h}{1 + m^2}$$

39. Cor. 2. By (20,) the equations

$$a x + b y + c = 0,$$

$$a x + b y + c' = 0$$

represent parallel lines.

Now, (38,) the respective perpendiculars from the origin are $p = -\frac{c}{\sqrt{a^2 + b^2}}, p' = -\frac{c'}{\sqrt{a^2 + b^2}}$

the difference of these gives the perpendicular distance of the said lines =

$$\frac{c - c'}{\sqrt{a^2 + b^2}}$$

Similarly the perpendicular distance between the parallel lines

$$y = m x + b,$$
\[ y = m \ x + h' \]

\[ V \]

is equal to

40. To express the distance between a given point \( x' \ y' \) on a given line and its intersection with another given line,

Let the equation of the given line, which passes through the point \( x' \ y' \), be

\[ a' (x - x') + b' (y - y') = 0; \]

and that of the other given line

\[ a \ x + b \ y + c = 0. \]

Let \( xy \) be their point of intersection and its ordinates will be the same in both equation. The latter one being put in the form

\[ a (x - x') + b (y - y') + (a' x + b' y + c) = 0, \]

we shall, by means of it and the former, find

\[ x - x' = \frac{b (ax' + by' + c)}{a'b' - b'a}, \quad y - y' = -\frac{a' (ax' + by' + c)}{a'b' - b'a} \]

Therefore, for the required distance,

\[ D = \sqrt{\left( x - x' \right)^2 + \left( y - y' \right)^2} = \frac{ax' + by' + c}{a'b' - b'a} \cdot \sqrt{a^2 + b^2}. \]
Otherwise,

Let $p$ be the perpendicular from the point $x'y'$ on the line $ax + by + c = 0$, then, $i$ being the angle of inclination of the lines, we obviously have

$$p = D \sin i$$

$$D = \frac{p}{\sin \sin i}$$

Hence, substituting the values of $p$ and $\sin i$ already laid down, (37,29,) we get

$$D = \frac{\pm(a'x' + b'y' + c)}{a'b - b'a}(a^2 + b^2).$$

41. Cor. When the equations are $y = mx + h$ for the intersected line and $y - y' = m'(x - x')$ for the line passing through the point $x'y'$, substitute $m, m'$ respectively for $a, a'$ and $-1$ for $b, b'$ and $h$ for $c$, (13) and

$$D = (m x' + h - y')/(m-m')(1+m^2).$$
SECTION IV.
TRANSFORMATION OF THE AXES.

42. With the view of expressing any particular lines or
curves, being the loci of points, by algebraic equations, we are
manifestly at liberty to assign to the origin and the axes any
positions whatever, relative to the said loci; and hence, when
the equation of a locus is complex it becomes some times use-
ful to assume another position of the axes which will reduce
it to a more simple form. This transformation, which is called
the transformation of co-ordinates, is effected by expressing
the original in terms of the new co- ordinates, of any point.
which will, of course, be ready for substitution in any equation
or formula, appertaining to the former axes, so as to produce
the equivalent involving the new co-ordinates. For these oper-
ations the three following propositions are necessary.

43. An expression involving the two rectangular co-or-
dinates being given to find the corresponding expression in
terms of the co-ordinates to when the origin is transferred to a given point, the axes retaining a parallel position.

Let $O'$ be the given point to which the origin is to be transferred; and let its position referred to the axes $OA, OB$ be $OG=a, GO'=b$; also

let the position of the point $P$ related to the new axes $O'A', O'B'$ parallel to $OA, OB$, be $x'y'$ viz $O'D'=x''$ and $D'P=y'$. Then is

$$OD=x=O'D'+OG=x'+a,$$
$$DP=y=D'+GO'=y'+b,$$

Which substituted for $x$ and $y$ will give the expression requires, wherein $a, b$, the ordinates of the new origin $O'$, will be given constants and $x', y'$ the ordinates of $P$ referred to the new axes.

By this means we transfer the origin $O$ to a point whose ordinates are $x=a, y=b$,

44. An expression involving the ordinates of a point referred to two rectangular axes being given to find the corresponding expression when the point is referred to two other rectangular axes mak-
ing a given angle with the former and proceeding from the same origin,

We shall omit the axis of $y$ in the figure for the sake of simplicity, since it is sufficient to bear in mind that the positive ordinates $y$ extend from the axis of $x$ upwards. Let $OA$ be the original

and $OA'$ the new axis of $x$; then are $A x$ upwards. Let $OA$ be the original and $OA'$ the new axis of $x$; then are $OD, DP$ and $OD', DP$ the co-ordinates of $p$. Draw $D'H$ perpendicular and $D'K$ parallel to $OA$; and let the ordinates $OD'$, $D'P$, which refer the point $P$ to the new axes be $x'y$. Then, assuming the given $\angle OA'OA = \angle D'PK = w$, the difference of which gives

\[ OD = x = x' \cos w - y' \sin w \ldots \ldots (1); \]

also $D'H = KD = x' \sin w$ and $PK = y' \cos w$, which added give

These values of $OD$ and $PD$ introduced instead of $x$ and $y$ will produce an expression involving $x'y'$ and the given angle $w$

It must be here observed that the new axis $OA'$ of $x$ is taken on that side of $OA$ on which the ordinates $y$ are positive; when
taken on the contrary or under side of $OA$, the angle $w$ will have a negative value.

45. An expression involving the co-ordinates of a point related to two rectangular axes being given to deduce the corresponding equivalent in terms of the ordinates of the same point referred to two other rectangular axes making a given angle with the former and having a different origin. Let $O'A'$ be the new axis of $x \cdot PD'$ perpendicular to it from the point $P$, and $O'$ a parallel to $OA$. Denote the position of the new origin $O'$ by $ab$, viz: $OG=a$, $GO'=b$; let the position of $P$ with respect to the new axes be $x'y'$, that is $O'D'=x', D'P=y'$; and, as before, denote the given angle of inclination $A'Oa$ by $w$. Then, (44,) the co-ordinates of $P$ referred to $O'a$ as an axis of $x$ are

$$O'd=x' \cos w - y' \sin w; \quad p \ d=x' \sin w + y' \cos w;$$

and hence, (43,) the values of the original ordinates are

$$OD=x=x' \cos w - y' \sin w + a...(1),$$

$$PD=y=x' \sin w + y' \cos w + b...(2).$$
By substituting these instead of $x$ and $y$ in the proposed expression we shall get an expression, involving $x'y'$ with the new additional constants $a$ $b$ and the given angle $w$, which will be the one required.

**SECTION V,**

**EQUATIONS OF THE SECOND DEGREE.**

46. The general form for equations of the second degree, being those in which the ordinates $x$ $y$ are involved to the second power, is

$$Ax^2+By^2+Cxy+ax+by+c=0$$

wherein each of the constants $A,B,C,a,b,c$, may be either positive or negative,

Let us in the first place transfer the equation to two other rectangular axes parallel to the original ones and having their origin at a point whose ordinates are $a$ $b$; and, (43,) by substituting $x+x'$ and $y+y'$ for $x$ and $y$, we shall find the corresponding equation to be
\[ A (x^2 + 2x'x + x'^2) + B (y^2 + 2y'y + y'^2) + c(x'y + y'x' + x'y') + a(x-x') + b(y+y') + c = 0; \]

which arranged for \( x \) and \( y \) becomes

\[ Ax^2 + By^2 + Cxy + (2Ax + Cy' + a)x + (2By' + Cx' + b)y + (Ax'2 + By'2 + Cx'y' + a x' + b y' + c) = 0. \]

47. The first three co-efficients \( A, B, C \) stand unaffected with the new constants \( x', y' \), by which we observe that they are independent of the position of the origin; and hence position of the origin of any equation of the second degree depends entirely on the values of the three last coefficients \( a, b, c \).

48. We may now assume the values of the two ordinates \( x'y' \) at pleasure since the position of the new origin is entirely arbitrary; and consequently, by the principles of algebra, we may fulfil any two possible conditions which involve them; let us therefore put the coefficients of \( x \) and \( y \) each equal to nothing, viz:

\[ 2A x' + C y' + a = 0, \]
\[ 2B y' + C x' + b = 0; \]
and thence

\[ x' = \frac{Cb - 2B a}{4(AB - C^2)}, \quad y' = \frac{C a - 2A b}{4AB - C^2}; \]

lience also, by substitution, the last term

\[ Ax'^2 + By'^2 + Cx'y' + ax'y' + by'y' + c = \frac{Cab - Ab^2 - Ba^2}{4AB - C^2} + C; \]

or by assuming

\[ Cab - Ab^2 - Ba^2 + c(4AB - C^2) = G, \]

it becomes

\[ G = \frac{4AB - C^2}{4AB - C^2} \]

The equation is thus transformed into

\[ A x'^2 + By'^2 + Cx'y' + 4 - G/4 = 0 \quad \text{(a)}, \]

in which the fourth and fifth terms are wanting.

49. Let us now transfer this equation to two other rectangular axes inclined at an angle \( w \) with the former and retaining the same origin; and,(44,) substituting \( x \cos w - y \sin w \) and \( x \sin w + y \cos w \) for \( x \) and \( y \), we get for the corresponding equation.
A \left( x^2 \cos^2 w + y^2 \sin^2 w - 2x y \cos w \sin w \right) + B \left( x^2 \sin^2 w + y^2 \cos^2 w + 2x y \cos w \sin w \right) + c \left( x^2 \cos w \sin w - y^2 \cos m \sin w + x y (\cos^2 w - \sin^2 w) \right) + G/(4 A B - C^2) = 0,

which arranged for \( x \) and \( y \), observing that

\[
\cos^2 w - \sin^2 w = s \cos^2 w \text{ and } 2 \cos w \sin w = \sin^2 w
\]

becomes

\[
\left( A \cos^2 w + B \sin^2 w + C \cos w \sin w \right) x^2 + \left( A \sin^2 w + B \cos^2 w - (\cos w \sin w) \right) y^2 + \left( \cos^2 w - (a-b) \sin^2 w \right) x y + \frac{G}{4 A B - C^2} = 0
\]

By taking the value of \( u \) so as to exterminate \( x \) \( y \),

\[
C \cos 2 w - (A - B) \sin 2 w = 0
\]

and \( \tan 2 w = \frac{A - B}{C} \),

which reduces the equation to

\[
A \cos^2 w + B \sin^2 w + C \cos w \sin w = A'' ,
\]

\[
A \sin^2 w + B \cos^2 w - C \cos w \sin w = B''
\]

and it hence appears that every line of the second order may be referred to two determinate rectangular axes so that its equation shall be transformed into the above form. By assuming

\[
A \cos^2 w + B \sin^2 w + C \cos w \sin w = A'' ,
\]

\[
A \sin^2 w + B \cos^2 w - C \cos w \sin w = B''
\]

it becomes
\[ A''x^2 + B''y^2 + G / (4A'B - C^2) = 0 \] ...(b).

50. Now if the principal semi-diameters of an ellipse and hyperbola be denoted by \(a', b'\), and the former be taken for the axis of \(x\) and the origin at the centre, their equations will be as follow:

For the ellipse
\[
\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \quad orb'^2x^2 + a'^2y^2 - a'^2b'^2 = 0;
\]

and for the hyperbola
\[
x^2/a'^2 - y^2/b'^2 = +1, \quad orb'^2 - a'^2y^2 + a'^2b'^2 = 0
\]

the under sign representing the conjugate hyperbola.

The signs may be all changed if necessary.

By means of these two equations and the foregoing transformed equation, (b,) we deduce the following particulars relative to the general equation.

51.1st When \(A'', B''\) are both negative and \(G, 4A'B - C^2\) have the same sign, the equation determines an *ellipse*; and when \(A'', B''\) are both of them positive and \(G \) and \(4A'B - C^2\) have different signs, the locus is also an *ellipse*.  

---

1 For the immediate values of \(A'', B''\) see article 73.
52. 2nd. When $A''$, $B''$ are of different signs and $G$ not $= 0$, the locus is *an hyperbola*.

53. 3rd. In each of these cases the squares of the principal semi-diameters are equal to

$$\frac{+G}{A''(4AB - C^2)} \frac{+G}{B''(4AB - C^2)}$$

the under sign being for the ellipse and either sign for the hyperbola.

54. 4th. The values of $G, A''B''$ are determined from the equations

\[ G = C a b - A b^2 - B a^2 + c (4 A B - C^2) \ldots \text{(l)}, \]

\[ \tan 2w = \frac{C}{A - B} \ldots \text{(2)}, \]

\[ A'' = A \cos^2 w + B \sin^2 w + C \cos w \sin w \]

\[ B'' = A \sin^2 w + B \cos^2 w - C \cos w \sin w \]

wherein $w$ is the angle included between the original axis of $x$ and the principal diameter of the curve.

55. 5th. The position of the centre of the curve is determined by
56. 6th. When the equation is of the form $Ax^2 + By^2 + Cxy + c = 0$ wherein the fourth and fifth terms of the general equation are wanting, we have $a = 0, b = 0$ and thence $x' = 0, y' = 0$ which therefore shews the origin to be at the centre of the curve. This agrees with equation (b,) article 49, where the origin is transferred to the centre.

57. 7th. By adding the equations (3), article 54, we find $A'' + B'' = A + B$.

Hence we see that, whatever be the position of the axes of co-ordinates, the sum of the co-efficient of $x^2$ and $y^2$ will be the same.

58. 8th. When $G = 0$ and also $A''$ and $B''$ of different signs, the general equation defines a straight line. For in this case the transformed equation (b), article 49, becomes

$$A''x^2 + B''y^2 = 0,$$

which gives
\[ \frac{y}{x} = -\frac{A''}{B''} \]

and this value is real when \( A'', B'' \) have different signs.

59. 9th. In the two following cases it will be found that no real values of \( x \) and \( y \) can possibly fulfil the equation (b); and consequently that the equation can have no locus. First. When \( G \) and 4 \( AB - C^2 \) are of the same sign and \( A'', B'' \) both of them positive Second. when \( G \) and 4 \( AB - C^2 \) are of different signs and \( A'', B'' \) are both negative

60. 10th. When \( G=0 \) and \( A'', B'' \) have the same sign, no real values of \( x \) and \( y \) can satisfy the equation (b,) except the particular case of \( x = 0, y = 0 \). In this case therefore the locus is the single point corresponding with the new origin \( x'y' \)

61. 11th. It appears that by changing the position of the origin to the centre \( x'y' \) the equation

\[ A x^2 + B y^2 + C x y + a x + b y + c = 0 \]

is transformed into the from

\[ A x^2 + B y^2 + C x y + h = 0, \]
wherein \( b = \frac{G}{4(4b - C^2)} \)

Also, that by taking two other axes of co-ordinates making an angle with these so that \( \tan 2w = c/(a-b) \), the equation \( Ax^2 + By^2 + Cxy + h = 0 \) becomes of the form \( A''x^2 + B''y^2 + h = 0 \) wherein \( A'' + B'' = A + B \) and the constant \( h \) is unchanged.

62. 12th. Let \( x'', y'' \) be the two semi-diameters of the curve \( Ax^2 + By^2 + Cxy + H = 0 \) which coincide with the axes co-ordinates to which it is referred, and they will be determined by taking first \( y = 0 \) and then \( x = 0 \) in the equation, the results being \( x''^2 = -h/(A) \), \( y''^2 = -h/(B) \).

Let also \( a', b' \) be the principal semi-diameters which coincide with the axes to which the equation

\[ A''x^2 + B'', y''^2 + h = 0. \]

appertains; and we similarly have

\[ a''^2 = -\frac{h}{A}, b''^2 = -\frac{h}{B} \]

Hence as \( A'' + B'' = A + B \), we have
\[ \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2} + \frac{1}{b^2} \]

That is the sum of the reciprocals of the squares of any two semi-diameters, of a curve of the second order, which are perpendicular to each other, is the same; and, in reference to the general equation, is

\[ \frac{A + B}{h} = \frac{A + B}{G} (4AB - C^2) \]

and 'thence

\[ \sin w \sqrt{A} - \cos w \sqrt{B} = \sqrt{A - jr - B}; \]

\[ (2Ax + Cy + a) \cos w + (2By' + Cx' + b) \sin w = - \frac{a\sqrt{B} - b\sqrt{A}}{\sqrt{(A + B)}} \]

and \( 2Ax + Cy' + a \sin w - (2By' + Cx' + b) \cos w = \)

\[ = 2\sqrt{A + B} \left\{ x' \sqrt{(A + y')} \sqrt{B} + \left( a\sqrt{A} + \frac{b\sqrt{B}}{2(A + B)} \right) \right\} \]

\[ 2x\sqrt{A(A + B)} + 2y'\sqrt{B(A + B)} + \frac{a\sqrt{A} + b\sqrt{B}}{\sqrt{A + B}} \]

= The equation thus becomes

\[ (A + B)y^2 - \frac{a\sqrt{B} - b\sqrt{A}}{\sqrt{A + B}} x \]

\[ -2\sqrt{A + B} \left\{ x' \sqrt{A + y'} \sqrt{B} + \left( a\sqrt{A} + b\sqrt{B} \right) \right\} \frac{1}{2(A + B)} y = 0 \]
65. We have, (63,) assumed \( x'y' \) to determine a point in the curve, but not restricted ourselves to any particular point; we may therefore take this point where the curve is intersected by a straight line whose equation is

\[
x\sqrt{A} + y\sqrt{B} + \frac{a\sqrt{A} + b\sqrt{B}}{2(A+B)} = 0
\]

\[
x'\sqrt{A} + y'\sqrt{B} + \frac{a\sqrt{A} + b\sqrt{B}}{2(A+B)} = 0
\]

by means of which we shall have

which reduces the equation to

\[
(A + B)y^2 - \frac{a\sqrt{B} - b\sqrt{A}}{\sqrt{A+B}} \cdot x = 0
\]

or

\[
y^2 - (a - b\sqrt{A})(A + B)^{(3/2)} \cdot x = 0 \quad \text{(C)}.
\]

But the equation of a parabola, whose parameter is \( p \) taking the origin at the vertex and the principal axis for the axis of \( x \), is

\[
y^2 = px \quad \text{or} \quad y^2 - p x = 0.
\]

Hence the following particulars:—

66. 1st. When \( a\sqrt{B} - b\sqrt{A} = 0 \) the locus is a \textit{Parabola} whose parameter is equal to

\[
n\sqrt{B} - b\frac{\sqrt{A}}{(A+B)^{3/2}}
\]
67. 2nd. According to article 12, the equation

\[
x\sqrt{A} + y + \sqrt{B} + \frac{a\sqrt{A} + b\sqrt{B}}{2(A+B)} = 0
\]

defines a straight line inclined to the original axis of x at an angle whose tangent = \(-\frac{\sqrt{A}}{\sqrt{B}}\) and Which is therefore equal to w, the inclination of the axis of the curve, with the axis of x; this line, (65,) also passing through the vertex xy, it must coincide with the axis of the curve. Therefore the above equation properly represents the principal diameter of the curve; by uniting it with the original equation we may hence find the co-ordinates \(x'y'\) of its intersection with the curve, or the vertex.

68. 3rd. If \(\sqrt{B} - b\sqrt{A} = 0\) or \(a\sqrt{B} = b\sqrt{A}\) the equation (c) gives simply

\[y = a,\]

which shews the locus in this case to be a straight line corresponding with the new axis of x, the equation of which is give,(67).
69. 4th. the equation \(4AB - C^2 = 0\) giving \(C^2 = \pm \sqrt{AB}\), the values of the constants \(A, B\) must have the same sign to make \(C\) real, that is, they must be either both of them positive or both negative; and hence we may consider them both positive for, when negative, they can be made so by preliminarily changing all the signs of the original equation. If, under this consideration, \(C\) be negative we shall have \(C = -2\sqrt{AB}\) instead of \(+2\sqrt{AB}\); in this case the foregoing operations hold good by either substituting \(-\sqrt{A}\) instead of \(\sqrt{A}\) or \(-\sqrt{B}\) for \(\sqrt{B}\), or by considering either \(\sqrt{A}\) or \(\sqrt{B}\) to have a negative value; and \(\tan w\) will become hence \(+\frac{A}{B}\) instead of \(-\frac{A}{\sqrt{B}}\).

Thus we see that, when \(C\) is negative, \(\tan w\) is positive and \(w < \frac{\pi}{2}\) and that, when \(C\) is positive, \(\tan w\) is negative and \(w < \frac{\pi}{2}\).

The foregoing investigations lead immediately to the solutions of the three following propositions:
70. To express the equations of the principal diameters of a curve of the second order which is determined by the general equation.

The co-ordinates of the centre, \( (55,) \) are

\[
x' = \frac{Cb - 2Ba}{4AB - C^2}, \quad y' = \frac{Ca - 2Ab}{4AB - C^2}.
\]

Let \( w \) denote the inclination of one of the principal diameters of the curve with the co-ordinate axis of \( x \); and, \( (54,) \)

\[
\tan 2 = \frac{C}{A - B},
\]

from which

\[
\tan \tan = \frac{\sec 2 - 1}{\tan 2} = \frac{\sqrt{\left\{ (A - B)^2 + C^2 \right\}} - (A - B)}{C}.
\]

Now the diameter being inclined to the co-ordinate axis of \( xx \) at the angle \( w \) and also passing through the centre \( x'y'x'y' \) of the curve, its equation, \( (22,) \) is

\[
y - y' = (x - x') \tan.
\]

Hence by substitution we have

\[
y - \frac{Ca - 2Ab}{4AB - C^2} = \left( x - \frac{Cb - 2Ba}{4AB - C^2} \right) \frac{\sqrt{\left\{ (A - B)^2 + C^2 \right\}} - (A - B)}{C} \quad ... (x),
\]

for the equation of the principal diameters.
The other diameter passing through the centre \( x'y'x'y \) perpendicular to this, its equation, (36,) is

\[
y - \frac{Ca - 2Ab}{4AB - C^2} = -\left(x - \frac{Cb - 2Ba}{4AB - C^2}\right) \cdot \frac{C}{\sqrt{\left(\left(A-B\right)^2 + C^2\right) - (A - B)}}
\]

or, which is the same,

\[
y - \frac{Ca - 2Ab}{4AB - C^2} = -(x - \frac{Cb - 2Ba}{4AB - C^2}) \cdot \sqrt{\left(\left(A-B\right)^2 + C^2 + (A + B)\right) + \frac{C}{C}} \quad \ldots(y)^2
\]

71. Cor. 1. If the origin of the ordinates be the centre of the curve its equation, (56,\(^2\)) will be of the form

\[
Ax^2 + By^2 + Cxy + c = 0;
\]

and we shall have \( a = 0, b = 0 \). In this case therefore the equations of the principal diameters are

\[
y = \frac{\sqrt{\left(\left(A-B\right)^2 + C^2\right) - (A - B)}}{C} \cdot x
\]

\[
andy = -\frac{\sqrt{\left(\left(A-B\right)^2 + C^2\right) + (A - B)}}{C} \cdot x
\]

2 By uniting these equations of the principal diameters with the given equation of the curve we may thence find the positions of the vertices.
72. *Note.* The equation

\[ \tan w = \frac{C}{A-B} \]

applies equally to both diameters. For, if \(2w\) fulfil this equation, it will also hold good when \(2w \pm p\) is substituted; and, \(w\) denoting the inclination of one of the diameters, \(\omega \pm \frac{\pi}{2}\) \(\omega \pm \frac{\pi}{2}\)

will evidently be that of the other.

From this equation we derive generally

\[ \tan = \frac{\sec 2w-1}{\tan 2w} = \pm \frac{\sqrt{((A-B)^2+C^2)-(A-B)}}{C}, \]

the upper sign appertaining to one of the axes and the under sign to the other.

Thus, by making use of the under sign, the equation \((x)(x)\)

will become the same as the equation \((y)(y)\), and vice versa

because

\[ \pm \sqrt{((A-B)^2+C^2)-(A-B)} \cdot \frac{\sqrt{((A-B)^2+C^2)-(A-B)}}{C} \]

\[ = -1. \]

When \(4AB - C^2 = o, 4AB - C^2 = o\), see article 67.
73. The equation of a curve of the second order being given to find the values of its principal semi-diameters.

The squares of the semi-diameters are, (53,) equal to

\[
\pm G \frac{A''(4AB - C^2)}{B''(4AB - C^2)},
\]

wherein, (54,)

\[
G = Cab - Ab^2 - Ba^2 + c(4AB - C^2);
\]

\[
A'' = A\cos^2 + B\sin^2 + C\cos\sin,
\]

\[
B'' = A\sin^2 + B\cos^2 - C\cos\sin,
\]

and \(\tan 2 = \frac{C}{A - B}\).

From the last we deduce

\[
\cos^2 = \frac{1}{2} \left(1 + \frac{1}{\sec 2w}\right) = \frac{1}{2} \left(1 + \frac{A - B}{\sqrt{(A - B^2) + C^2}}\right),
\]

\[
\sin^2 = \frac{1}{2} \left(1 - \frac{1}{\sec 2w}\right) = \frac{1}{2} \left(1 - \frac{A - B}{\sqrt{(A - B^2) + C^2}}\right),
\]

\[
\cos w\sin w = \frac{C}{2\sqrt{(A - B^2) + C^2}};
\]

and hence we get

\[
A'' = \frac{A + B + \sqrt{(A - B)^2 + C^2}}{2}.
\]
\[ B'' = \frac{A + B + \sqrt{\left\{(A - B)^2 + C^2\right\}}}{2} \]

These and the foregoing value of G substituted, the squares of the principal semi-diameters of the curve are found equal to

\[ \pm \frac{2\{Cab - Ab^2 - Ba^2 + c\left(4AB - C^2\right)\}}{(4AB - C^2)\left[A + B + \sqrt{\left\{(A - B)^2 + C^2\right\}}\right]} \]

\[ \pm \frac{2\{Cab^2 - Ab^2 - Ba^2 + c\left(4AB - C^2\right)\}}{(4AB - C^2)\left[A + B + \sqrt{\left\{(A - B)^2 + C^2\right\}}\right]} \]

the under sign being for the ellipse and either sign for the hyperbola, (53.)l

74. When the origin is at the centre of the curve, (61,)
\[ a = o, b = o; a = o, b = o; \] and therefore in this case the squares of the principal semi-diameters are equal to

\[ \pm \frac{2c}{A + B - \sqrt{\left\{(A - B)^2 + C^2\right\}}} , \quad \pm \frac{2c}{A + B - \sqrt{\left\{(A - B)^2 + C^2\right\}}} \]
75. To determine the particular description of a curve of the second order from the immediate relative values of the constants which belong to its equation.

In (51), (52) and the subsequent articles, the different cases are severally stated, throughout the various relations of $A'', B''; G, 4AB − C^2$, &c., where $A'', B''$ are, (54,) expressed in terms of coefficients $A, B, C$, by means of the arc $w$ as a subsidiary. ‘It is’ hence only necessary to transfer the relations of $A'', B''$ to those of the immediate coefficients $A, B, C$, which may be easily effected from their values which have already been found, (73,) viz:

$$A = \frac{A + B + \sqrt{(A − B)^2 + C^2}}{2},$$

$$A = \frac{A + B + \sqrt{(A − B)^2 + C^2}}{2},$$

Thus it is evident that, when $(A + B)^2$ is greater than $(A − B)^2 + C^2$ the sign of $A+B$ cannot be affected with either the addition or subtraction of $\sqrt{(A − B)^2 + C^2}$, cones
quently that the values of $A''B''$ will both have the same sign with $A+B$. But, when $(A+B)^2$, is greater than $(A-B)^2+C^2$, we shall have $(A+B)^2$ is greater than $4AB - C^2$ positive. Hence, when $4AB - C^2$ is positive, $A''$ and $B''$ will both of them have the same sign with $A+B$, that is, they will both be positive when $A+B$ is positive and both negative when $A+B$ is so.

It is also pretty obvious that, when $(A+B)^2$ is less than $(A-B)^2+C^2$, the values of $A''B''$ will have different signs, that is, the one will be positive and the other negative.

In this case we shall have $(A+B)^2 - (A-B)^2+C^2=4AB - C^2$ negative. Thus we see, when $4AB - C2$ is negative, that $A''$, $B''$ are of different signs.\(^3\)

Again, under the class $4AB - C^2 = 0$, when the value of $a(b(A=0)$, we shall have $2(A(a(B−b(A)=0$

$$=2a (AB − 2 Ab=0$$

Or $Ca-2Ab=0$

\(^3\) These relations are also pretty evident from the equations $A''+ B'' = A + B$, $4A''B'' = 4AB - C$. 
Hence also, when $a(B - b(A \neq 0)$, we shall have $Ca - 2Ab$ not $-0$.

By carefully comparing these relations with the articles (51), (52), (58), (59), (60), (66), and (68), we find the descriptions of the curve to be as in the following arrangement, wherein

$$G = Cab - A b^2 - B a^2 + c(4AB - C^2).$$

SECTION VI.

FORMULAE FOR CURVES, &c. INVOLVING THE DIFFERENTIAL AND INTEGRAL CALCULUS.

76. The equation of a variable straight line being given to find the point of intersection of two of its positions which are indefinitely near to each other,

Let $y = m x + h$ be the equation of the line.

It is plain that when $m$, $h$ are given constants the straight line is given and fixed; thus, in order that the line may assume another position it is necessary that one or both of characters $m, h$ shall become of different values. Hence, under the circumstances of the proposition one or both of the values $m$, $h$ must be subject to variation,
Let \( x' y' \) be the co-ordinates of the required point which will, of course, fulfil the equation

\[ y' = mx' + h. \]

If we now suppose the lines to an indefinitely near consecutive position, the point of intersection \( x' \) and \( y' \) being fixed during the change its ordinates \( x' \) and \( y' \) will hence remain invariable. Hence differentiating, and considering \( x'y' \) as constant we get

\[ 0 = x' dm + dh, \]

from which and the above equation we find the required point to be

\[ x' = -dh / dm, y' = h - m dh / dm. \]

To compute these values it is necessary for \( h \) to be a function of \( m \); this is, in fact, necessary to impose a law on the variation of the line, the position of which would otherwise be absolutely arbitrary.
77. cor.1. If the line be supposed in motion this point will obviously be the center of instantaneous rotation; and its locus will evidently be that curve to which the line is always a tangent4* (see 81)—hence the nature of this curve may be found by eliminating the introduced variable from the values of x’ and y’.

78. Cor. 2. Similarly to the foregoing may we find the point of intersection of two indefinitely near positions of a variable curve by differentiating its equation and considering x’, y’ constant; the values of x’, y’ being determined from the given and the resulting equation.

79. To find the area of a curve comprehended between two given values of y and the axis of x.

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4 This is rendered evident by considering it inversely; thus, by supposing a tangent to move over a curve line its successive indefinite intersections will obviously coincide with the points of “Contact and therefore trace out the same curve.
By taking two values of $y$ indefinitely near to each other, the space included between them may obviously be considered as rectangular and consequently as having a value $= ydx$.

Thus we have

The area $= \int ydx$.

This integral, between the limits $x'y'$, $xy$ will give the area contained between the ordinates $y'$ and $y$.

80. To find the length of any portion of a curve from the equation between its rectangular co-ordinates.

Let $s$ denote the length of the curve corresponding with the ordinates $x'y$ and reckoned from any given point, and we shall evidently have

$$ds^2 = dx^2 + dy^2;$$

$$ds = (dx^2 + dy^2)^{1/2};$$

and $s = \int \sqrt{(dx^2 + dy^2)}$

By taking this integral between the limits $x'y'$ and $xy$ we find the length of the portion of the curve intercepted by those points.
81. Def. Let $BPC$ be any curve. Then a straight line $RS$, drawn so as to touch it at any point $P$ is called a tangent; and the point $P$ is called the point of contact. Another straight line $PH$, drawn perpendicular to the curve at the point $P$ and consequently also perpendicular to the tangent $RS$, is called a normal at that point.

By the length of a tangent we generally understand that portion which is limited by its intersection with the axis of $x$ and the point of contact; the value of the normal is similarly understood to be that portion of it which is intercepted by the axis of $x$ and the point $p$. Thus, $PT$ is the length of the tangent and $PN$ that of the normal. OD and DP being the ordinates of $P$, TD is called the subtangent and DN the subnormal.

82. The equation of a curve being given to find the equation of the tangent at any point. Let $x'y'$ be the co-ordinates of the point of contact and $w$ the inclination of the tangent with the axis of $x$; then, (22) the equation is $y - y' = (x - x') \tan w$; but we obviously have $\tan w = \frac{dy'}{dx}$. The required equation is
\[ y - y' = \frac{dy'}{dx'} (x - x') \]

or \((x - x') - \frac{dx'}{dy'} (y - y') = 0,\)

wherein \(x'y'\) is the point of contact and \(xy\) any point whatever in the tangent.

Or it may be expressed differentially

\[ dy' (x - x') - dx' (y - y') = 0. \]

83. cor 1 Hence the equation to the normal through the same point, (36) is

\[ y - y' = -\frac{dx'}{dy'} (x - x') \]

or \((x - x') + dy'/dx' (y - y') = 0.\)

This may also be expressed differentially

\[ dx' (x - x') + dy' (y - y') = 0. \]

84. Cor, 2. The equation of the tangent at the point \(x'y'\)being

\[ dy' (x - x') - dx' (y - y') = 0 \]

or \(dy' x - dx' y + (y'dx' - x'dy') = 0,\)

the perpendicular drawn to it from the origin, (38,) is equal to
\[ x' \, dy' - \frac{y' \, dx'}{\sqrt{(dx'^2 + dy'^2)}} = + \frac{-x' \, dy' - y' \, dx'}{ds} \]

85. Cor, 3. Similarly, the perpendicular from the origin upon the normal is equal to

\[ x' \, dx' + y' \, dy = \frac{-x' \, dz'}{ds} \]

86. Cor, 4. By taking \( y=0 \) in the equation of the tangent, \( x \) will determine its intersection with the axis of \( x \) and hence in this case \( x'-x \) is the subtangent at the point \( x'y' \).

\[ \therefore \text{Subtangent} = x' - x = \frac{y' \, dx'}{dy'} \]

By similarly taking \( y=0 \), in the equation of the normal, we find

The subnormal = \( x'-x = \frac{y' \, dy'}{dx'} \).

Hence also

the Tangent = \( \left( y'^2 + \frac{(y'^2 \, dx'^2)}{dy'^2} \right) = \frac{y' \, d8'}{dx'} \),

and the Normal = \( \left( y'^2 + \frac{(y'^2 \, dy'^2)}{dx'^2} \right) = \frac{y' \, d8'}{dx} \)
$'$ being the arc of the curve corresponding with the point $x'y'$.

**ASYMPTOTES.**

87. Two curves or a curve and straight line are said to be asymptotic when they continually approach nearer and nearer to each other but do not meet at any finite distance.

By an asymptote to a curve we generally understand a straight line such that if it and the curve be indefinitely continued they will continually approach each other but never meet; or it may be considered as a tangent to the curve when the point of contact is at an infinite distance.

In the equation of the tangent let $y = 0$ and we shall find the intercept of the axis of $x$, between the origin and the tangent at the point $x'y'$, to be

$$x = x' - \frac{y'dx'}{dy'} = \frac{x'dy' - y'dx'}{dy'}.$$

Again by taking $x=0$ we similarly find the intercept of the axis of $y$, between the origin and the tangent, to be
If, when \( x' = 8 \) or \( y' = 8 \), either of these values of \( x \) and \( y \) are finite, the curve has asymptotes which will thence be determined.

When \( x \) is finite, but \( y \) infinite, the asymptote is parallel to the axis of \( y \).

When \( y \) is finite, but \( x \) infinite, the asymptote is parallel to the axis of \( x \).

But when the values of \( x \) and \( y \) are both of them infinite, the asymptote is at an infinite distance from the origin. In this case the curve is said to have no asymptote.

When the values of \( x \) and \( y \) are both \( 0 \) the asymptote passes through the origin and its position must be determined from the value of \( \frac{dy}{dx} \), when \( x' = 8 \) or \( y' = \infty \).

88. The equation of a curve being given to find whether, at at particular point, it is convex or concave to the axis of \( x \).
As in article 82, let \( x'y' \) denote the ordinates of the proposed point in the curve and \( w \) the inclination of the tangent at that point with the axis of \( x \) and

\[
\frac{dy'}{dx} = \tan w.
\]

Now when the curve at the point \( x'y' \) is concave to the axis of \( x \) the angle \( w \) will evidently, if \( x \) increase, decrease when \( y \) is positive and increase when \( y \) is negative; and therefore \( d \frac{dy''}{dx} \) will have a sign contrary to that of \( y \). But when the curve is convex to the axis of \( x \) the inclination of tangent will obviously, when \( x \) increases, increase or decrease accordingly as \( y \) is positive or negative and consequently \( d \frac{dy'}{dx} \) will have the same sign with \( y'y \).

Hence, taking \( dX' \) constant the curve at the point \( x'y' \) will be concave towards the axis of \( x \) when \( d^2 y' \) has a contrary sign with \( y' \) and it will present a convex side towards the axis of \( x \) when \( d^2 y' \) has the same sign with \( y' \). Or, which amounts to the same, the curve is convex or concave to the axis of \( x \) accordingly as \( y'd^2 y' \) is positive or negative.
89. Cor. 1. Hence also, by supposing $dy'$ constant, the curve at the point $x'\; y'$ will be convex or concave to the axis of $y$ accordingly as $d^2 x'$ has the same or a different sign with $x'$; or it will be convex or concave towards the axis of $y$ accordingly as $x' \; d^2 x'$ is positive or negative. 90. Cor. 2. When a curve is first convex and becomes afterwards concave to the axis of $x$ it must have passed a point of contrary flexure ____________ in this case, supposing $dx'$ constant, $d^2 y'$ will hence experience a change of sign; and the point of contrary flexure will evidently be where $d^2 y' = 0$.

91. The equation of a curve being given to find the radius of curvature, or the radius of that circle which touches it most intimately at any given point.

A tangent to any curve may be conceived to be a straight line drawn through two of its points which are indefinitely near to each other; and hence the first differentials of the ordinates which appertain to the tangent must correspond with those of the curve at the point of contact.
Similarly may we conceive the osculating circle or the circle of curvature to be that circle which passes through three successive points of the curve which are indefinitely near to each other; in this case therefore, both the first and second differentials of the ordinate which belong to the circle and curve must correspond at the point of contact.

Let \(x'' \ y'\) be the co-ordinates of the centre of the circle and we shall have \(x-x'', \ y-y''\) for the two lines drawn from it respectively parallel to \(x\) and \(y\) and terminating in the circumference at the point of contact; hence, denoting its radius by its equation, (25,) is

\[
(x-x')^2 + (y-y')^2 = \delta^2
\]

Now since, as has been observed, this circle corresponds with the curve at two other points contiguous to the point of contact, we may differentiate twice and consider the first and second differentials of the ordinate \(xy\) as agreeing with those of the curve. Hence differentiating, observing that \(x''y'\) are invariable, we get
\[ dx(x'' - x') + dy(y'' - y') = 0, \]

Where
\[ d^2x(x'' - x') + d^2y(y'' - y') + ds^2 = 0; \]

being the length of the curve.
\[ ds^2 = dx^2 + dy^2 \]

From these two equations we deduce
\[ x - x' = \frac{-dy}{dyd^2x - dx^2y} ds^2, \quad y - y'' = \frac{dx}{dyd^2x - dx^2y} ds^2 \]

Hence we find
\[ s^2 = (x - x')^2 + (y - y'')^2 = ds^6 / (dyd^2x - dx^2y)^2 \]
\[ = +ds^3 / (dyd^2x - dx^2y) \]

In this expression for the radius of curvature we may assume an independent variable at pleasure.

92. It may be otherwise very clearly determined by conceiving the centre of the circle of curvature to be the intersection of two normals drawn from two points of the curve which are indefinitely near to each other. Let a normal be drawn from the point, intercept of the curve between these point being ds. Let
also two tangents be drawn at these points, the former making an angle \( w \) with the axis of \( x \). Then the angle \( \pm dw \) included by the tangents will evidently be the same as that included by the normals; and as the normals, which are radii of the osculating circle, subtend the arc \( ds \) of the curve, we obviously have

\[
\pm s \ dw = ds \\
\text{and } = - + \frac{ds}{dw}
\]

But

\[
tan w = \frac{dy}{dx}
\]

\[
dw = \frac{d \ tan w}{1 + \tan 2w} = \left( \frac{d}{dx} \frac{dy}{dx} \right) = (dx^2 \cdot \frac{d}{ds^2} \left( \frac{dy}{dx} \right))
\]

Therefore by substitution

\[
= \pm \frac{ds^3}{dx^2 \cdot d \left( \frac{dy}{dx} \right)} = \pm \frac{ds^3}{dy^3 x - dxd^2 y}
\]

93. Since

\[dx^2 + dy^2 = ds^2\]

we have, differentiating '.
\[ \frac{dx}{ds} = \frac{dy}{dy^2x - dx^2y} \]

and hence
\[ = \frac{ds^3}{dy^2x - dx^2y} \]

94. By taking \( x \) for the independent variable, or supposing \( dx \) to be constant,
\[ \frac{dx}{ds} = \frac{-dy}{dy^2x - dx^2y}, \quad y - y' = \frac{dxds^2}{dy^2x - dx^2y} \]

95. From the equations
\[ x - x' = \frac{-dy}{dy^2x - dx^2y}, \quad y - y' = \frac{dxds^2}{dy^2x - dx^2y} \]
we find the position of the centre of the circle of curvature to be

\[ x' = x + \frac{dy ds^2}{dy'^2 x - dx'^2 y} \]

\[ y' = y + \frac{dy ds^2}{dy'^2 x - dx'^2 y} \]

or, supposing \( dx \) constant,

\[ x' = x - \frac{dy ds^2}{dx'^2 y} \]

\[ y' = y + dx \frac{ds^2}{dx'^2 y} \]

By means of these and the equation of the curve, the ordinates \( xy \) and their differential may be eliminated; and an equation will thence be found expressing the relation between \( x'' \) and \( y'' \), which will hence define the locus of the centre of the osculating circle for all points in the curve. This locus is denominated the *evolute* of the curve; and, on the contrary, the curve is called its *involute*.

96. As the centre of the osculating circle may be conceived to be the point of intersection of two normals which are
in-definitely near to each other, it is obvious that the normal at any point of the curve must be a tangent to the evolute See article 77.

Thus we see that a tangent drawn to the evolute at any point coincide with the radius of the osculating circle which is drawn to ‘the point contact.

97. The equation of this tangent, \((62,)\) gives
\[
dy'' (x - x'') - dx'' (y - y'') = 0.
\]

Let us now differentiate the equation
\[
(x - x')^2 + (y - y')^2 = s^2
\]
supposing \(x'' y''\) to vary, and we have
\[
(dx - dx'')(x - x'') + (dy - dy'')(y - y'') + sds;
\]

but, \(x'' y''\) appertaining to the normal of the curve at the point \(xy\), we have by its equation
\[
dx(x - x'') + dy(y - y'') = 0,
\]
which rejected and the signs changed, we get ‘
\[
dx''(x - x'') + dy''(y - y'') = -d.
\]
From this and the preceding equation of the tangent to the evolute we find

\[ x - x' = -sds \cdot \frac{dx'}{ds^2}, \quad y - y'' = -sds \cdot \frac{dy'}{ds^2}, \]

wherein

\[ ds''^2 = dx''^2 + dy''^2 \]

\( s'' \) being the arc of the evolute from any given point. These values of \( x - x' \) and \( y - y'' \) substituted in the equation

\[ (x - x')^2 + (y - y'')^2 = 2 \]

Give

\[ \delta^2 \cdot d \delta^2 / ds''^2 = \delta^2 \]

\[ ds''^2 = ds^2 \]

\[ ds'' = \pm d \]

The integral of this gives

where \( d' \) is the radius of curvature corresponding with the given point from which \( s' \) is estimated.
Hence we find the length of the arc of the evolutes between any two points to be equal to the difference between the radii of the corresponding osculating circles.

98. By means of this principle we discover that the curve may be described by the unwinding of an inextensible thread from off the evolute. Thus let \( p \) be any point in the curve and \( pp" \) the normal or radius of curvature touching the evolute at the point \( p" \); then, this line \( pp" \) being conceived to be a thread extending round the evolute, it is obvious, from the above property, that by unwinding this thread, keeping \( pp" \) always stretched, the point \( p \) will trace out the curve.

Considering the evolute as a curve, its involute is thus described.

99. It appears, from the foregoing, that any given curve can have but one evolute, but may have an indefinite number of involutes as the value of \( pp" \) at any point \( p" \) is indeterminate. Hence, for any particular involut, the value of \( pp" \) must be known at a given point \( p" \).
100. The evolute being given, the equation of its involutes may be found by means of the values of $x'', y''$, (95,) in terms of $xy$ and their differentials.

radius vector and the polar angle and conversely. This may be effected by means of the two following propositions.

105. To reduce any expression or formula, involving the rectangular co-ordinates $xy$ of any point and their differentials, into one involving the radius vector and the polar angle.

By taking the axis of $x$ for the polar axis, and the origin for the pole, we shall obviously have

$$x = r \cos \varphi, \quad y = r \sin \varphi;$$

and hence also, by differentiation,

$$dx = dr \cos \varphi - r d\varphi \sin \varphi,$$

$$dy = dr \sin \varphi + r d\varphi \cos \varphi;$$

$$d^2x = d^2r \cos \varphi - 2dr \ d\varphi \sin \varphi - rd^2\varphi \cos \varphi - rd^2\varphi \sin \varphi,$$

$$d^2y = d^2r \sin \varphi + 2dr \ d\varphi \cos \varphi - r dQ^2 \sin \varphi + r d^2\varphi \cos \varphi;$$
These values substituted in the given expression, will produce its equivalent in terms of $r$, $\phi \phi$ and their differentials.

By supposing $d\phi$ to be constant, the terms wherein $d^2 \phi$ occur will disappear,

106 If it be required to have the pole at a given point $x' y'$, we may previously transfer the origin of the rectangular axes to that point, by articles 43. Or we may substitute

$$x = r \cos \cos \phi + x', y = r \sin \sin \phi + y'$$

and $x'y'$ being constant, the values of $dx, dy, d^2 x, d^2 y$, &c. as above.

And, if the polar axis be required to make an angle $w$ with the axis of $x$, we must obviously substitute $\phi + w$ instead of $\phi$

107. \textit{To reduce any expression, involving the radius vector and the polar angle, into one involving rectangular co-ordinates.}
By taking the polar axis for the axis of $x$, and the pole for the origin of $xy$, we shall have, from the foregoing equations,

$$r = \sqrt{x^2 + y^2};$$

Also

$$\cos \varphi = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}, \sin \varphi = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}},$$

$$\tan \varphi = \frac{y}{x} \& c.,$$

or

$$\varphi = \cos^{-1} \frac{x}{\sqrt{x^2 + y^2}} = \sin^{-1} \frac{y}{\sqrt{x^2 + y^2}} = \tan^{-1} \frac{y}{x} \& c.,$$

wherein $\cos^{-1} \frac{x}{\sqrt{x^2 + y^2}}$ signifies the arc whose cosine is $\frac{x}{\sqrt{x^2 + y^2}}, \& c.$.

The required transformation may be accomplished by the substitution of these values; and the origin may afterwards be transferred to any given point, (43).

108. *The polar equation of a curve being given to find, the length of any arc of it.*

By referring the points of the curve to rectangular co-ordinate axes, we have, (80,)
\[ ds^2 = dx^2 + dy^2. \]

Hence, substituting the values of \( dx, dy \), (105,) we get

\[ ds^2 = dr^2 + r^2 d\varphi^2 \]

\[ ds = \sqrt{dr^2 + r^2 d\varphi^2} \]

\[ s = \int \sqrt{dr^2 + r^2 d\varphi^2} + \text{const.}, \]

the value of the constant being such as to make the complete integral vanish at the point whence the arc is estimated.

109. To find the perpendicular from the origin on the tangent at any given point.

The equation of the tangent at any given point \( x'y' \), (82,) is

\[ dy'(x' - x') - dx'(y' - y') = 0, \]

or, \[ dy'.x - dx'.y - (x' \ dy' - y' \ dx') = 0 . \]

Let \( p \) be the required perpendicular, and, as In article 84, we shall hence have

\[ p = \frac{x' dy' - y' dx'}{\sqrt{(dy' 2 - dx' 2)}} = \pm \frac{x' dy' - y' dx'}{ds'} \]

This reduced, for the polar equation, (105,) gives
110. To find the sectoreal area continued between the curve and any two radii vectors.

Let’ us imagine two radii vectores infinitely near to each other, containing the indefinitely small angle $dx$, and sub-tending the element $ds$ of the curve. The sectoreal element thus formed by these radii vectors and $ds$ may obviously be considered as a plane triangle; and the perpendicular from the origin on the opposite side $ds$ will obviously correspond with that on the tangent. Therefore, $p$ denoting this perpendicular, we shall hence have the area of this sectoral element $= \frac{pds}{2}$.

That is,

$$d. (\text{Sectoreal Area}) = \frac{pds}{2}$$

But, (109,)

$$p = \frac{\frac{x\,dy - y\,dx}{ds}}{ds} = \frac{r^2\,d\varphi}{ds}$$

$$d.(\text{SectorealArea}) = \frac{x\,dy - y\,dx}{2^2} = \frac{r^2\,dq}{2^2}$$

$$\text{Sect.Area} = \frac{x\,dy - y\,dx}{2} + \text{const} = \frac{yr^2\,dq}{2} + \text{const}.$$
The value of the constant must be determined from the position of the radius vector from which the area is to be computed.

111. **To express the inclination of the tangent to a curve, with the radius vector.**

Let the required angle under the tangent and radius vector be $Tr$, and $p$ the perpendicular on the tangent $T$. Then is

\[
\sin Tr = \frac{p}{r}
\]

\[
\cos Tr = \frac{\sqrt{r^2 - p^2}}{r}, \quad \tan Tr = \frac{p}{\sqrt{r^2 - p^2}}
\]

Therefore, substituting the value of $p$, (109,)

\[
\sin Tr = \frac{rd\phi}{ds} = \frac{rd\phi}{\sqrt{d^2 + r^2 d\phi^2}},
\]

\[
\cos Tr = \frac{dr}{ds} = \frac{dr}{\sqrt{dr^2 + r^2 d\phi^2}},
\]

\[
\tan Tr = \frac{rd\phi}{dr}.
\]

Any of these formulae will serve for the determination of the required angle.
112. To express the polar subtangent in terms of \(r\) and \(\phi\).

The polar subtangent is the straight line drawn from the pole perpendicular to the radius vector and terminating in the tangent. Since

\[
tanTr = \frac{rd\phi}{dr},
\]

we have

\[
Subtangent = rtan Tr = \frac{r^2 d\phi}{dr}.
\]

113. Cor. Sometimes the equation of a curve is expressed between the radius vector and the perpendicular on the tangent. In this case we may make use of

\[
tanTr = P / \sqrt{(r^2 - p^2)},
\]

which gives

\[
Subtangent = \frac{rP}{\sqrt{r^2 - p^2}}.
\]

114. To express the radius of curvature of a curve in terms of the radius vector and the polar angle.

We have, (92,)

\[
s = \pm \frac{ds^3}{dydx^2x - dx^2y}
\]

Now, pursuing the suppositions used in article 105, we get
\[ dyd^2x - dx^2y = \]
\[ dr (\sin \varphi d^2x - \cos \varphi d^2y) + r d\varphi (\cos \varphi d^2x + \sin \varphi d^2y) \]
\[ = -dr (2dr \ d\varphi + rd^2 \varphi) + r d\varphi (d^2r - rd'^2) \]
\[ = -d\varphi (r^2 d\varphi^2 + 2dr^2 - rd^2r) - r drd^2 \]

Hence
\[ S = \pm \frac{ds^3}{d\varphi (r^2 d\varphi^2 + 2dr^2 - rd^2r) + r drd^2 \varphi} \]

By supposing \( \varphi \) to vary independently
\[ S = \pm \frac{ds^3}{d\varphi (r^2 d\varphi^2 + 2dr^2 - rd^2r) = \pm \frac{(dr^2 + r^2 d\varphi^2)^3}{2}} \]

115. The equation of a curve between the radius vector and the corresponding perpendicular on the tangent being given, to find the angle contained by any two radii vectors.

By (109),
\[ p = \frac{r^2 d\varphi}{\sqrt{dr^2 + r^2 d\varphi^2}} \]
This solved for \( d\varphi \) gives
\[ d\varphi = \frac{p dr}{r \sqrt{r^2 - p^2}} \]
the integral of which between the proposal limits will give the angle sought.

116. Given the equation between the radius vector and the perpendicular on the tangent, \( F(r, p) = 0 \), to find the length of any arc of the curve, and also the sectoreal area between any two radii vectors.

It has been found, (115), that

\[
d\phi = \frac{pdr}{r \sqrt{r^2 - p^2}}.
\]

This substituted for \( ds \) in (108) (110), we find

\[
ds = \frac{rdr}{\sqrt{r^2 - p^2}}, s = \frac{rdr}{\sqrt{r^2 - p^2}} + \text{const};
\]

and the sectoreal Area = \( \frac{prdr}{2 \sqrt{r^2 - p^2}} + \text{const} \).

117. To express the radius of curvature in terms of the radius vector and the perpendicular on the tangent.

By (109),

\[
p = \frac{r^2 d\phi}{\sqrt{dr^2 + r^2 d\phi^2}}.
\]
Let $d\rho$ be supposed constant and

$$dp = rdr\ d\rho \frac{r^2d\rho^2 + 2dr^2 - rd^2r}{\left(dr^2 + r^2d\rho^2\right)^3/2}$$

But, (114,)

$$s = \pm \frac{\left(dr^2 + r^2d\rho^2\right)^3/2}{d\rho\left(r^2d\rho^2 + 2dr^2 - rd^2r\right)}$$

$$s\ dp = \pm rdp$$

$$s = \pm \frac{rdr}{dp}.$$

118. The equation of a curve between the radius vector and the perpendicular on the tangent being given, to find the similar equation for its evaluate.

The radius of curvature of the curve at any point $rp$ coincides with the normal and touches the evaluate, (96). Let $R, p$ be the radius vector and the perpendicular on the tangent which belong to the evaluate at the point of contact. By drawing the figure it will be at once perceived that $p$ and $p$ constitute a rectangle with the tangent and normal of the curve; also that
\[ P^2 = r^2 - p^2, \]
\[ R^2 = (s - p)^2 + p^2, \]

which inverted, and the value of \( P \) by the former substituted in the latter, we get
\[ r^2 - p^2 = P^2, \]
\[ (s - p)^2 + r^2 - P^2 = s^2 - 2ps + r^2 = R^2 \]

The value of \( s \) being previously determined, (117,). we can by means of these and the given equation of the curve, \( f (r, \rho) = 0 \), eliminate \( r \) and \( \rho \), which will produce the equation wanted.

119. Let \( R' \) and \( P' \) be the radius vector and the perpendicular on the tangent which belong to an involute of the curve; as the curve is its evolute we have from the foregoing

\[
\left( R' \frac{dR'}{dP'} - P' \right)^2 + R'^2 - P'^2 = R^2.
\]
The values of \( pr \) by these equations substituted in the equation of the curve we shall find an equation involving \( R' P' \) and their differentials. If it can be integrated the equation of the involutes of the curve will thence be found.

END OF PART FIRST.